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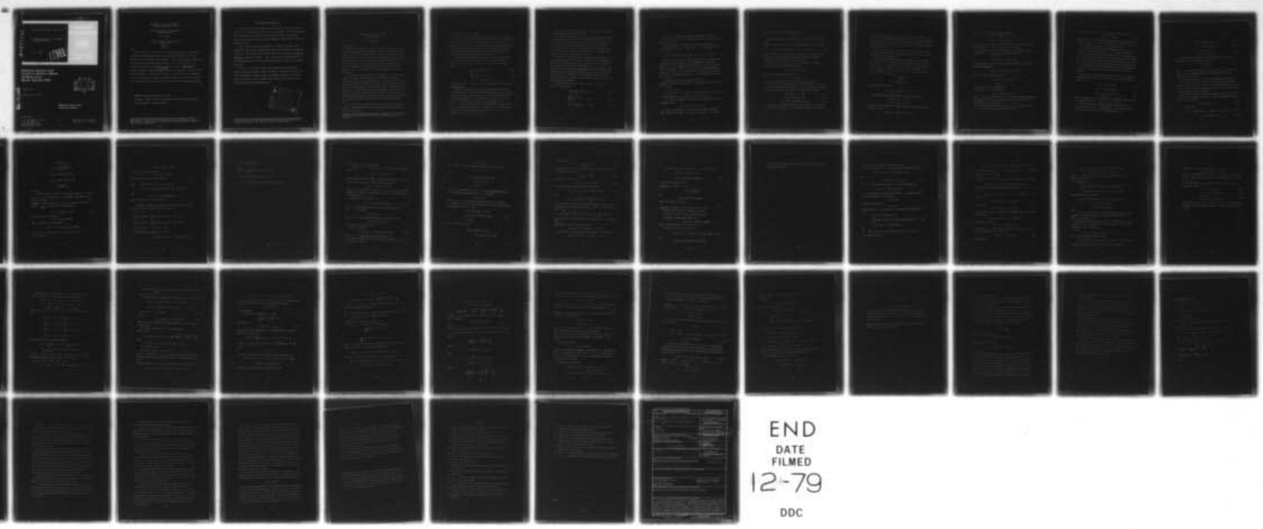

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NUMERICAL STABILITY IN AN INVERSE SCATTERING PROBLEM

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ABSTRACT

The main result of this paper is a stability theorem for a certain class of difference algorithms designed to give approximate solutions of a model inverse scattering problem in one dimension. This stability result guarantees the convergence of the approximate solutions to the exact solution of the problem as the grid of the difference scheme is refined. ~~We present~~ the results of numerical experiments are presented based on one of these schemes, in which second-order convergence is observed. Furthermore the cost (that is, the dependence on N of the number of arithmetic operations required to compute the solution at N grid points) of the algorithms discussed below is essentially optimal.

AMS(MOS) Subject Classification: 35R30

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SIGNIFICANCE AND EXPLANATION

Many problems of applied mathematics involve the inference of mechanical properties of a medium, parts of which are not accessible to direct observation, from measurements of scattering of small-amplitude waves. Such problems arise in exploration geophysics, physical chemistry, and ultrasound tomography, among other areas.

Many of these problems are equivalent in principle to boundary value problems for certain partial differential equations. Any method of approximate solution for such boundary value problems must have the attribute of <u>numerical stability</u>, in order to be useful: that is, the round-off errors present in all computation must not lead to uncontrolled growth in errors of the computed quantities.

In this paper, a class of difference algorithms for a simple model inverse scattering problem is shown to be stable. This result implies that these algorithms will actually compute approximate solutions to the inverse scattering problem. In fact, explicit error bounds can, in principle, be extracted from the results presented here.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

NUMERICAL STABILITY IN AN INVERSE SCATTERING PROBLEM

W. W. Symes

§1. Introduction

The main result of this paper is a stability theorem for a certain class of difference algorithms designed to give approximate solutions of a model inverse scattering problem in one dimension. This stability result guarantees the convergence of the approximate solutions to the exact solution of the problem as the grid of the difference scheme is refined. We present the results of numerical experiments based on one of these schemes, in which second-order convergence is observed. Furthermore the cost (that is, the dependence on N of the number of arithmetic operations required to compute the solution at N grid points) of the algorithms discussed below is essentially optimal.

The algorithms of this paper are difference approximations to a certain hyperbolic boundary value problem. In a previous paper [1], we showed that this hyperbolic boundary value problem, the solution of which leads to a solution of the above-mentioned inverse scattering problem, is equivalent to a certain Volterra integral equation, and the latter was solved constructively, with estimates. Roughly the same plan is followed in this paper. An approximate version of the Volterra equation is first solved, with estimates. This discrete Volterra equation is then shown to be almost equivalent to a certain difference approximation to the hyperbolic boundary value problem. The relation is close enough that stability estimates for a class of difference approximations follow.

It is interesting that the usual approaches to proving stability of difference schemes for time-dependent problems (see for instance [2]) do not seem to apply to the problem considered here.

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§2. Statement and Discussion of Results

We first discuss the continuum problem and its solution, with various estimates, as presented in [1]. The continuum results provide the major tools for the approximate solution, as well as the form of the estimates which must be reflected in the stability theorem for the various difference schemes. We then present some notational conventions (concerning difference schemes and uniform estimates) to which we will adhere throughout. Finally, we state the difference schemes which form the main subject of the paper, and outline their stability properties.

The inverse scattering problem of this paper is: given a real-valued function $F : [0, 2T] \to \mathbb{R}$, find a real-valued $V : [0, T] \to \mathbb{R}$ and $H \in \mathbb{R}$ so that $F(t) = U(0, t)$, $0 < t \le 2T$, for the solution U of the initial boundary-value problem

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V(x)\right) U(x, t) = 0 \tag{2.1a}$$

$$\left(\frac{\partial U}{\partial x} + HU\right)(0, t) = 0 \qquad t > 0 \tag{2.1b}$$

$$U(x, 0) = \delta(x), \; \frac{\partial U}{\partial t}(x, 0) \equiv 0 \tag{2.1c}$$

in the region $\{(x, t) : 0 \le x \le T, 0 \le t \le 2T\}$. We shall refer to this problem as <u>the inverse problem</u>.

This inverse problem can be considered a very simple instance of the inverse scattering problem for a mechanical continuum supporting small-amplitude wave propagation. Roughly speaking, you are required to construct a vibrating one-dimensional continuum having equations of motion of the form (2.1a), boundary condition of the form (2.1b), and having a prescribed response (back-scattered wave at $x = 0$) $F(t)$, $t > 0$, to an impulsive ("broad band") incident disturbance (initial conditions (2.1c)), for a prescribed duration $2T > 0$. The geometry of such a hyperbolic mixed problem shows clearly that the boundary value $F(t) = U(0, t)$ for $0 < t \le 2T$ depends on the coefficient $V(x)$ only for $0 \le x \le T$.

This problem is typical in various respects of a large number of time-dependent and time-independent inverse scattering problems.

Various other problems which may be treated by the techniques described below are discussed briefly in the Conclusion (section 8). Furthermore, at least in the limit $T \to \infty$, this problem has a long and illustrious history, being a time-dependent version of the inverse spectral problem for Sturm-Liouville operators solved by I. M. Gel'fand and B. M. Levitan in their seminal paper [3]. The connection between the half-axis version of the inverse problem for (1a,b,c) and the Gel'fand-Levitan inverse spectral problem is explained in [1], section 3.

It was shown in [1] that solution of the inverse problem is equivalent to solution of a hyperbolic boundary value problem ((2.2) below). The Russian mathamatician Chudov seems to have been the first to suggest the possible use of this boundary value problem as a means for solving inverse problems (see [4], final section). We shall therefore call this problem the Chudov problem. As we shall explain elsewhere, the approach to inverse problems through the Chudov problem is closely related to the recent work of Deift and Trubowitz [5] on the 1-dimensional Schrödinger inverse scattering problem, and to the work of Hochstadt, Hald, and Levitan ([6], [7], [8]) on inverse Sturm-Liouville problems.

The Chudov problem is as follows. Let $C_T = \{(x, t) : 0 \le x \le \min(t, 2T - t)\}$. Find $W : C_T \to \mathbb{R}$ such that

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V(x)\right) W(x, t) = 0 \qquad (2.2a)$$

$$\begin{aligned} &W(0, t) = F(t) \\ &\left(\frac{\partial W}{\partial x} + HW\right)(0, t) = 0 \qquad 2T \ge t \ge 0 \\ &H = F(0) \end{aligned} \qquad (2.2b)$$

$$V(x) = -2\frac{d}{dx} W(x, x) \qquad 0 \le x \le T \qquad (2.2c)$$

Here, as in the inverse problem, the datum of the problem is the function $F : [0, 2T] \to \mathbb{R}$. Note that (2.2) is nonlinear, by virtue of the coupling of solution and coefficient in (2.2c).

In our previous work [1], estimates are given for various Sobolev norms of W, V in terms of similar norms of F, and a number $\epsilon > 0$ with the property that, for any $\phi \in L^2 [0, T]$,

$$\int_0^T |\phi|^2 + \int_0^T ds \int_0^T dt\ \phi(s)\ \bar{\phi}(t)\ (\tfrac{1}{2}\ (F(s + t) + F(|s - t|)) \geq \epsilon \int_0^T |\phi|^2 \tag{2.3}$$

The results of [1] are phrased in terms of the spaces $W^{m,1}$ of functions with n absolutely integrable derivatives. Another somewhat unusual Sobolev Space, denoted by $\mathcal{W}^{m,1}$ in [1], intervenes as an auxiliary device. This space is, roughly speaking, the subspace of $W^{m,1}(\mathbb{R}^2)$ in which each function has a well-defined restriction to each line in $\mathbb{R}^2$, lying in $W^{m,1}$ of functions on the line, and the restrictions vary continuously with the choice of line. Precise definitions are found in [1], section 2.

The following result is proved in sections 5, 6 of [1]:

Theorem 1. Problem (2a,b,c) has a solution W in the space $\mathcal{W}^{2,1}(C_T)$ if and only if F satisfies

(i) $F \in W^{2,1}([0, 2T])$

(ii) there exists $\epsilon > 0$ so that (2.3) is satisfied for all $\phi \in L^2([0, T])$.

If these are satisfied, then $V \in W^{1,1}([0, T])$ and is given by (2.2c).

It is convenient to paraphrase condition (ii) of the theorem as follows. Define the symmetric kernel

$$G(s, t) = \frac{1}{2}\ (F(s + t) + F(|s - t|))$$

If F satisfies (i), the kernel G is continuous, hence defines a bounded self-adjoint Hilbert-Schmidt operator G on $L^2([0, T])$. Denote by I the identity operator on $L^2([0, T])$. Then (ii) can be stated:

$$I + G \geq \epsilon > 0 \tag{ii'}$$

The method of solution of the Chudov problem (2.2) is based on the following result:

Theorem 2. A function $W \in W^{2,1}(C_T)$ solves the Chudov problem (2) if and only if

$$G(s, t) = W(s, t) + \int_0^s dy\, W(y, s)\, W(y, t) \quad \text{for} \quad (s, t) \in C_T . \tag{2.4}$$

Remark 1) We shall refer to (2.4) as the G-L equation, or (GL). It appears first in the paper of Gel'fand and Levitan [3], and also expresses the group law of propagation of initial values for the mixed problem (1a,b).

2) for $(s, t) \in C_T$, we have $0 \leq s \leq t$. If we associate to the kernel W (which is at least continuous) the Volterra operator

$$W\, \phi(y) = \int_y^T dt\, W(y, t)\, \phi(t)$$

then you can write the GL equation in the form

$$I + G = (I + W)^{+}(I + W)$$

which clearly illustrates the necessity of hypothesis (ii') of Theorem 1.

The GL equation allows the derivation of a number of estimates for V in terms of F in various norms other than the Sobolev $(m, 1)$-norms of the (sharp) stability statement. For our purposes the following C^0-estimates will be sufficient.

$$\|W\|_\infty \leq \|F\|_\infty (1 + \epsilon^{-1} T \|F\|_\infty) \tag{2.5a}$$

$$\|D_2 W\|_\infty \leq \|DF\|_\infty \exp (T \|W\|_\infty) \tag{2.5b}$$

$$\|V\|_\infty \leq 2 \|DF\|_\infty + \|W\|_\infty (\|W\|_\infty + 2T \|D_2 W\|_\infty) \tag{2.5c}$$

At this point, it should be mentioned that the estimate (2.5) is not really a stability result, since the problem (2.2) is nonlinear. However it is very easy to prove genuine stability results (that is, local Lipschitz estimates for the map $F \mapsto V, H$) on the basis of boundedness results like (2.5). For both the continuum and approximate problems, therefore, we will refrain from stability statements (which are necessary, for instance, to derive explicit error bounds) and present only boundedness results like (2.5).

This concludes our discussion of the (continuous) inverse problem. We now turn to approximate methods of solution, and begin by establishing notation and terminology for the difference schemes we shall use.

The approximate algorithms of this paper are difference schemes for computing certain grid functions. The grids are uniform and rectangular, and the granularity will be denoted by Δ. We will use the same letters for the discrete approximants as for their continuous counterparts; thus, $F(n)$ is meant to approximate $F(n\Delta)$, $W(n, m)$ corresponds to $W(n\Delta, m\Delta)$, etc. The number of (linear) gridpoints will be N. We will have need of the sup norms

$$\|F\| = \sup_{1 \le n \le N} |F(n)|$$

or

$$\|W\| = \sup_{1 \le n, m \le N} |W(n, m)|$$

and occasionally the ℓ^p_Δ norms

$$\|F\|_p = \left(\sum_{n=1}^{n} \Delta \, |F(u)|^p \right)^{1/p} \qquad 1 \le p < \infty$$

The basic difference operators defined on grid functions are given by

$$D^+ F(n) = \Delta^{-1}(F(n + 1) - F(n))$$

$$D^- F(n) = \Delta^{-1}(F(n) - F(n - 1))$$

The partial difference operators on 2-dimensional grid functions will be denoted by subscripts, for instance:

$$D_1^+ W(n, m) = \Delta^{-1}(W(n + 1, m) - W(n, m))$$

and so on. We will also need the diagonal derivatives

$$D_n^+ W(n, n) = \Delta^{-1}(W(n+1, n+1) - W(n, n))$$

$$D_n^- W(n, n) = \Delta^{-1}(W(n, n) - W(n-1, n-1))$$

Our aim is to produce estimates for various difference schemes, based on such grids, which are uniform in Δ as $\Delta \to 0$, that is, with $T = N\Delta$ fixed.

The first set of estimates pertain to solutions of an approximate GL equation. We suppose that $F : [0, 2T] \to \mathbb{R}$ is a Lipschitz function, and set

$$F(n) = F(n\Delta)$$

$$G(n, m) \equiv \frac{1}{2}\{F(n+m) + F(|n-m|)\}$$

If we approximate the integral in (GL) by right-endpoint Riemann sums, we obtain the discrete Volterra equation

$$G(n, m) = W(n, m) + \sum_{k=1}^{n} \Delta\, W(k, n)\, W(k, m) \tag{2.6}$$

for $1 \leq n < m$. We can require (harmlessly)

$$\begin{aligned} G(n, n) &= 2W(n, n) + \sum_{n=1}^{n} \Delta\, W(k, n)^2 \\ W(n, m) &= 0 \quad , \qquad n > m \end{aligned} \tag{2.7}$$

In section 4, we obtain the following results:

1) The system (2.6 - 2.7) can be written

$$\frac{1}{\Delta} I + G = \Delta(\frac{1}{\Delta} I + W)^{\dagger} (\frac{1}{\Delta} I + W)$$

where I is the $N \times N$ identity matrix, and thus represents the Cholesky decomposition of the matrix $\frac{1}{\Delta} I + G$. A solution therefore exists if and only if the L.H.S. is positive-definite, with lower bound $\epsilon > 0$.

2) Under suitable conditions, which we shall not discuss here, the solution W of (2.6) - (2.7) converges in sup norm to the solution of (2.4).

3) The solution W of (2.6) - (2.7) is estimated by (Proposition 4.1)

$$\|W\| \leq \|F\| + \epsilon^{-1} \|F\|_2^2 \tag{2.8}$$

Further, provided Δ is small enough in relation to $\|F\|$, one can also bound the partial differences of W by entire functions of $\|W\|$ and norms of differences of F, which are linear in F near $F = 0$ (Propositions 4.2, 4.3).

It is evident from (2.8) that uniform estimates as the grid is refined $(\Delta \to 0)$ will only be achieved if the lower bound $\epsilon = \epsilon(\Delta)$ eventually stabilizes. Our results therefore apply when the discrete approximations to the backscattered wave have been chosen so as to guarantee such a uniform lower bound. We will not discuss methods for extracting such discrete grid functions from experimental data in this paper, although this very interesting matter should certainly be considered further.

Various elementary estimates and identities, used to prove the above-mentioned estimates and for various other purposes, are collected in section 3 for easy reference.

The identities and estimates of section 3 are used in section 5 to compute a difference scheme satisfied by the solution W of (2.6) - (2.7). The result is (equation (5.14)):

$$(D_2^+ D_2^- - D_1^+ D_1^-)W + VW = \Delta R \tag{2.9a}$$

$$V(n) = -(D_n^- + D_2^+ + D_1^-)W(n, n) \tag{2.9b}$$

$$\left.\begin{aligned} D^+ W(0, m) + HW(0, m) &= \Delta B(m, \Delta) \\ W(0, m) &= F(m) \end{aligned}\right\} \tag{2.9c}$$

Here R (2-dimensional) and B (1-dimensional) are grid functions whose sup norms are bounded in terms of $\|F\|$, $\|D^{\pm} F\|$, $\|D^+ D^- F\|$, T, Δ, and ϵ.

This result is useless for computation purposes, since the R.H.S. of (2.9a), (2.9c) depend explicitly on W in a complicated way. On the other hand, (2.9) is clearly related to the following difference approximation to (2.2):

$$(D_2^+ D_2^- - D_1^+ D_1^-)W_0 + V_0 W_0 = 0 \tag{2.10a}$$

$$V_0(n) = -(D_n^- + D_2^+ + D_1^-)W_0(n, n) \tag{2.10b}$$

$$\begin{aligned} W_0(0, m) &= F(m) \\ D^+ W_0(0, m) + H W_0(0, m) &= 0 \end{aligned} \tag{2.10c}$$

The main result of this paper is the following estimate, proved in section 7; relating the solutions of the systems (2.9) and (2.10):

$$\begin{aligned} \|W - W_0\| &\leq \Delta C_1 \\ \|V - V_0\| &\leq \Delta C_2 \end{aligned} \tag{2.11}$$

Here C_1, C_2 are entire functions of $\|F\|$, $\|D^\pm F\|$, $\|D^+ D^- F\|$, T, ϵ^{-1}, and Δ. (In fact, C_1, C_2 are exponential polynomials; and can be written out explicitly, although we do not do so here).

The estimates (2.8) and (2.11) combine to yield boundedness statements like (2.5) for the solution of (2.10). An interesting difference is that the resulting estimates on $\|V_0\|$ involve, in the limit $\Delta \to 0$, the modulus of Lipschitz continuity of the derivative DF, whereas the sup norm estimate for V in the continuum problem (2.5c) only requires that DF be continuous.

The system (2.10) is still inefficient for computation; as it has local truncation error (on the boundaries) of first order. A second-order-consistent approximation to (2) is, for instance,

$$(D_2^+ D_2^- - D_1^+ D_1^-)W + VW = 0 \tag{2.12a}$$

$$\begin{aligned} W_1(0, m) &= F(m) \\ W_1(1, m) &= \left(\frac{1}{2} V(0)\Delta^2 - H\Delta\right)W(0, m) + \frac{1}{2}(W(0, m+1) + W(0, m-1)) \end{aligned} \tag{2.12b}$$

$$V(n) = -(D_n^+ + D_n^-)W(n, n) \tag{2.12c}$$

A bonus of our method is that the stability of the 1st-order-accurate system (2.10) implies the stability of the 2nd-order-accurate system (2.12). Indeed, the two differ by terms which are $O(\Delta)$, which may be shifted to the R.H.S. Our arguments are exactly designed to extend the estimates for (2.10) to similar estimates for (2.12).

Well-known arguments then guarantee that the discrete coefficient V, constructed by solving (2.12) on a machine, differs from the exact solution of (2.2), evaluated at the corresponding grid points, by an error proportional to Δ^2. The constant of proportionality can even be estimated in terms of sup norms of differences of F, T, ϵ^{-1}, and the round-off characteristics of particular machines, though we shall not do this here.

Instead, we give the results of some numerical experiments based on the second-order scheme (2.12). These are displayed in section 7. We end with a discussion, in section 8, of related problems which may be solved by the same methods, as well as the relation of our results to previous work on inverse scattering problems.

§3. Elementary Identities and Estimates

We begin with some estimates for discrete Volterra equations (obtained from Volterra equations by replacing integrals with right-end-point Riemann sums). These are entirely elementary, but we know of no reference which states exactly the estimates we need, so we give complete proofs. We end with a number of useful identities from the calculus of finite differences.

<u>Lemma</u> (3.1) Suppose that, for $1 \le n \le N$

$$g(n) = \phi(n) + \sum_{k=1}^{n-1} \Delta\, W(n, k)\phi(k)$$

Then

$$\|\phi\|_\infty \le \|g\|_\infty \exp(N\Delta \|W\|_\infty)$$

<u>Proof</u>. Define $T\psi(n) = g(n) - \sum_{k=1}^{n-1} \Delta\, W(n, k)\psi(n)$ for any grid function $\{\psi(n), 1 \le n \le N\}$. Then we seek a fixed point of T. The point is, of course, that the fixed point exists and is globally asymptotically stable. For instance, set

$$\phi_0 \equiv 0$$

$$\phi_{p+1} = T\, \phi_p$$

and

$$u_p = \phi_p - \phi_{p-1}$$

Then $u_{p+1} = T_0\, u_p$, where

$$T_0\, \psi(n) = -\sum_{u=1}^{n-1} \Delta\, W(n, k)\psi(k)$$

Claim:

$$|u_p(n)| \le \frac{(n\Delta \|\tilde{W}\|_\infty)^p}{p!} \|g\|_\infty \quad p = 1, 2, \cdots$$

The claim certainly holds for $p = 1$. Suppose that it holds up to $p - 1$. Then

$$|u_p(u)| = \left| -\sum_{k=1}^{n-1} \Delta\, W(n, k)u_{p-1}(k)\right|$$

$$\leq \Delta \|W\|_\infty \sum_{k=1}^{n-1} |u_{p-1}(k)|$$

$$\leq \Delta \|W\|_\infty \sum_{k=1}^{n-1} \frac{(k\Delta \|W\|_\infty)^{p-1}}{(p-1)!} \|g\|_\infty$$

$$= \frac{1}{(p-1)!} (\Delta \|W\|_\infty)^p \sum_{k=1}^{n-1} k^{p-1} \|g\|_\infty$$

$$\leq \frac{1}{(p-1)!} (\Delta \|W\|_\infty)^p \int_0^n dk\, k^{p-1} \|g\|_\infty$$

$$= \frac{(n\Delta \|W\|_\infty)^p}{p!} \|g\|_\infty$$

as claimed.

It follows that $\sum_{p=0}^{\infty} u_p$ converges. Of course $\sum_{p=0}^{\infty} u_p = \lim_{p\to\infty} \phi_p = \phi$ is a fixed point of T, which is clearly unique, and the estimates on u_p add up to the estimates of the Lemma. q.e.d.

Corollary 3.2. Suppose that for each n ; $1 \leq n \leq N$, we have $1 + \Delta W(n, n) > 0$, and set $\rho = \sup_{1\leq n\leq N} |(1 + \Delta W(n, n))|^{-1}$. Then the equation

$$g(n) = \phi(n) + \sum_{k=1}^{n} \Delta W(n, k)\phi(k) \tag{3.1}$$

has a unique solution which satisfies

$$\|\phi\|_\infty \leq \rho \|g\|_\infty \exp(\rho N\Delta \|W\|_\infty)$$

Proof. The equation (3.1) can be rewritten

$$g(n) = (1 + \Delta W(n, n))\phi(n) + \sum_{k=1}^{n-1} \Delta W(n, k)\phi(k)$$

Set $\psi(n) = (1 + \Delta W(n, n))\phi(n)$, $\tilde{W}(n, k) = (1 + \Delta W(n, n))^{-1} W(n, k)$.

Then ψ satisfies

$$g(n) = \psi(n) + \sum_{k=1}^{n-1} \Delta \tilde{W}(n, k)\psi(k)$$

hence according to the previous lemma,

$$\|\psi\|_\infty \leq \|g\| \exp (N\Delta \|\tilde{W}\|_\infty)$$

which immediately implies the asserted estimate.

q.e.d.

Lemma Suppose $\{U(n, m) : 1 \leq n \leq m \leq N\}$ satisfies

$$G(n, m) = U(n, m) + \sum_{n=1}^{n-1} \Delta W_1(k, n)U(k, m) + \sum_{k=1}^{n-1} \Delta U(k, n)W_2(k, m)$$

Then

$$\|U\| \leq \|G\| \exp [N\Delta \max (\|W_1\|, \|W_2\|)]$$

Proof. Exactly parallel to the proof of Lemma 1.

q.e.d.

We conclude with a number of "summation by parts" formulas which will be used in section 5. The proofs are all trivial, so we omit them.

1. $D^+ D^- = D^- D^+$

2. $\sum_{k=1}^{n-1} \Delta(D^+ f(k))g(k) = - \sum_{k=1}^{n-1} \Delta f(k)D^- g(k) + f(n)g(n-1) - f(1)g(0)$

3. $\sum_{k=1}^{n-1} \Delta(D^- f(n))g(k) = - \sum_{k=1}^{n-1} \Delta f(k) (D^+ g(k)) + f(n-1)g(n) - f(0)g(1)$

4. $D^- f(k) = D^+ f(k-1)$, $D^- f(k+1) = D^+ f(k)$

5. $\sum_{k=1}^{n-1} \Delta(D^- D^+ f(k))g(k) = \sum_{k=1}^{n-1} \Delta f(k) (D^- D^+ g(k))$

$$+ D^- f(n)g(n) - f(n)D^- g(n) - D^- f(1)g(1) + f(1)D^- g(1)$$

6. Set $F(n) = \sum_{k=1}^{n-1} \Delta f(k, n)$

Then

6a. $D^- F(n) = \sum_{k=1}^{n-1} \Delta D_2^- f(k, n) + f(n - 1, n - 1)$

6b. $D^+ F(n) = \sum_{k=1}^{n-1} \Delta D_2^+ f(k, n) + f(n, n + 1)$

6c. $D^+ D^- F(n) = \sum_{k=1}^{n-1} \Delta D_2^+ D_2^- f(k, n) + D_n^- f(n, n) + D_2^+ f(n, n)$

§4. Estimates for Cholesky Decompositions

Let $\{F(n), n = 0, \cdots, N\}$ be an $(N + 1)$ - vector, and define the symmetric array G by

$$G(n, m) = \frac{1}{2} [F(n + m) + F(|n - m|)]$$

Suppose that the matrix $\left[\frac{1}{\Delta} I + G\right]_{n,m=1}^{N}$ is positive-definite, where I is the $N \times N$ identity matrix. Then $\frac{1}{\Delta} I + G$ admits a Cholesky decomposition

$$(\frac{1}{\Delta} I + G) = \Delta (\frac{1}{\Delta} I + W^{\dagger}) (\frac{1}{\Delta} I + W) \tag{4.1}$$

where W is a triangular array, $W(n, m) = 0$ if $n > m$. In this section we obtain estimates for W and its partial differences in terms of F and its differences, and the lower bound for $\frac{1}{\Delta} I + G$.

The formula (4.1) can be rewritten

$$G(n, m) = W(n, m) + \sum_{k=1}^{n} \Delta W(k, n)W(k, m) \tag{4.2}$$

for $m > n$. This suggests the introduction of the Hilbert subspaces $H_m \subset H = \ell^2_\Delta(1, \cdots, N)$ defined by

$$\psi \in H_m \Leftrightarrow \psi(n) = 0 \quad , \quad n \geq m$$

Let $\Pi_m : H \to H_m$ be the orthogonal projections. Then (4.2) can be written

$$\Pi_m G_m = \Pi_m \Delta (\frac{1}{\Delta} + W^{+}) \Pi_m W_m \tag{4.3}$$

(here $G_m(n) = G(n, m)$, $W_m(n) = W(n, m)$)

From (4.3) you obtain

$$\| \Pi_m G_m \|_2 = \langle W_m \ , \ \Pi_m \Delta (\frac{1}{\Delta} + W) \Pi_m (\frac{1}{\Delta} + W^{\dagger}) \Pi_m W_m \rangle \tag{4.4}$$

where $\langle , \rangle$ is the inner product associated with the norm $\| \ \|_2$.

We claim that the operator in the R.H.S. of (4.4) is invertible. In fact, it is of the form $K K^{\dagger}$, where

$$K = \Pi_m (\frac{1}{\Delta} + W)\Pi_m$$

is an operator on H_m. Notice that the triangularity of W implies that

$$\Pi_m W \Pi_m = W \Pi_m$$

Thus

$$K^\dagger K = \Delta \Pi_m (\frac{1}{\Delta} + W^\dagger)\Pi_m \Pi_m (\frac{1}{\Delta} + W)\Pi_m$$

$$= \Delta \Pi_m (\frac{1}{\Delta} + W^\dagger)(\frac{1}{\Delta} + W)\Pi_m$$

$$= \Pi_m (\frac{1}{\Delta} + G)\Pi_m$$

Since $\frac{1}{\Delta} + G$ is bounded below on H by $\epsilon > 0$, it is <u>a fortiori</u> bounded below by ϵ when sandwiched between projections on H_m. It follows that

$$\|K \psi\|^2 \geq \epsilon \|\psi\|^2$$

for all $\psi \epsilon H_m$. Then K must also be invertible, hence $K K^\dagger$ is invertible.

Since both $K^\dagger K$ and $K K^\dagger$ are invertible on H_m, they have the same spectrum, which necessarily lies above ϵ. Thus

$$(W_m, \Pi_m \Delta (\frac{1}{\Delta} + W)\Pi_m(\frac{1}{\Delta} + W^\dagger)\Pi_m W_m)$$

$$= (\Pi_m W_m, [\Pi_m \Delta (\frac{1}{\Delta} + W)\Pi_m(\frac{1}{\Delta} + W^\dagger)]\Pi_m W_m) \tag{4.5}$$

$$\geq \epsilon' \|\Pi_m W_m\|_2^2$$

If $n < m$, then $W_n = \Pi_m W_n$ and

$$(W_n, W_m) = \sum_{n=1}^{n} \Delta W(k, n)W(k, m)$$

$$= (\Pi_m W_n, \Pi_m W_m) = (\Pi_n W_n, \Pi_m, W_m) \tag{4.6}$$

On the other hand

$$\| \Pi_m G_m \|_2^2 \leq \| F \|_2^2 \tag{4.7}$$

By conbining (4.4), (4.5), (4.6), and (4.7) and using the Cauchy-Schwartz inequality one obtains finally that for $n < m$

$$\langle W_n, W_m \rangle \leq \epsilon^{-1} \| F \|_2^2 \tag{4.8}$$

and so from (4.2) that for $n < m$

$$|W(n, m)| \leq |G(n, m)| + \epsilon^{-1} \| F \|_2^2 \tag{4.9}$$

To estimate the diagonal elements $W(n, n)$, write from (4.1)

$$G(n, n) = 2W(n, n) + \Delta\, W(n, n)^2 + \sum_{k=0}^{n-1} \Delta\, W(k, n)^2$$

Thus

$$W(n, n) = \pm \frac{1}{\Delta} \sqrt{1 + [G(n, n) - \sum_{k=0}^{n-1} \Delta\, W(k, n)^2]\Delta} - \frac{1}{\Delta}$$

In order to maintain the positivity of the diagonal elements of $\frac{1}{\Delta} I + W$, as is required of the Cholesky decomposition, we must choose the $+$ sign in front of the radical. Since

$$\left| \sqrt{1 + [G(n, n) - \sum_{k=\phi}^{n-1} \Delta\, W(k, n)^2]\Delta} - 1 \right| \leq \left| G(n, n) - \sum_{k=\phi}^{n-1} \Delta\, W(k, n)^2 \right| \Delta$$

Another application of (4.8) shows that (4.9) is valid also for $n = m$. So one has proved

Proposition 4.1. The Cholesky factor W satisfies

$$\| W \|_\infty \leq \| F \|_\infty + \epsilon^{-1} \| F \|_2^2 \leq \| F \|_\infty (1 + \epsilon^{-1} N \Delta \| F \|_\infty)$$

The next step is to estimate the partial differences of W. We start with

$$D_2^+ W(n, m) = \frac{1}{\Delta} [W(n, m + 1) - W(n, m)] \quad , \quad n \leq m$$

Now

$$D_2^+ G(n, m) = \frac{1}{2\Delta} \cdot (F(n + m + 1) + F(|n - m - 1|) - F(n + m) - F(|n - m|))$$

$$= \frac{1}{2} [D^+ F(n + m) + D^+ F(|n - m|)]$$

$$= D_2^+ W(n, m) + \sum_{k=1}^{n} \Delta W(k, n) D_2^+ W(k, m) \tag{4.10}$$

A similar expression holds for $D_2^- W$

Proposition 4.2. Suppose $\rho > 1$ and

$$\Delta \leq \frac{\rho - 1}{\|F\|_\infty + \epsilon^{-1} \|F\|_2^2}$$

Then

$$\|D_2^\pm W\|_\infty \leq \rho \|D^\pm F\|_\infty \exp (\rho N\Delta \|W\|_\infty)$$

Proof. Apply Corollary 3.2 to (4.10).

q.e.d.

Proposition 4.3. Suppose ρ, Δ are as in Proposition 4.2. Then:

(a) $\|D_1^\pm W\| \leq \|W\|^2 + \|D^\pm F\| \times [1 + \rho N\Delta \|W\|_\infty \exp (\rho N\Delta \|W\|)]$

(b) $|D_n W(n, n)| \leq \|W\|^2 + \|D^\pm F\| [\frac{1}{2} + \rho N\Delta \|W\| \exp (\rho N\Delta \|W\|)]$

(c) $\|D_\mu^\pm D_\nu^\pm W\| \leq 2 \|W\| \max (\|D_1^\pm W\|, \|D_2^\pm W\|, \|D_n W\|)$

$$+ \|D^+ D^- F\| [1 + \rho N\Delta \|W\| \exp (\rho N\Delta \|W\|)]$$

Proof. According to formula (6) of section 3, for $n < m$

$$D_1^+ G(n, m) = \frac{1}{2} (D^+ F(n + m) - D^- F(|n - m|))$$

$$= D_1^+ W(n, m) + W(n + 1, n + 1) W(n + 1, m) + \sum_{k=1}^{n} \Delta (D_2^+ W(k, n)) W(k, m)$$

Hence

$$|D_1^+ W(n, m)| \leq \|D^\pm F\| + \|W\|^2 + N\Delta \|W\| \|D_2^+ W\|$$

which, together with Propositions 4.1 and 4.2, gives $(a)^+$. Similar computations give the other estimates.

§5. Partial Difference Equation for Cholesky Decomposition

As in the previous section, W will denote the lower triangular solution to

$$G(n, m) = W(m, n) + W(n, m) + \sum_{k=1}^{n} \Delta\, W(k, n)W(k, n) \tag{5.1}$$

where

$$G(n, m) = \frac{1}{2}\{F(n + m) + F(|n - m|)\} \tag{5.2}$$

For $n < m$, the first summand on the R.H.S. of (5.1) can be dropped, thus:

$$G(n, m) = W(n, m) + \sum_{k=1}^{n} \Delta\, W(k, n)W(k, m) \quad , \qquad n < m \tag{5.3}$$

For the remainder of the paper, we use the symbol $\Diamond$ to abbreviate the usual five-point approximation to the wave operator:

$$\Diamond \equiv D_2^+ D_2^- - D_1^+ D_1^-$$

An easy calculation shows that

$$\Diamond\, G(n, m) \equiv 0$$

Therefore, applying $\Diamond$ to both sides of (5.3) and using formula 6 (from section 3) gives

$$\begin{aligned} 0 = {} & W(n, m) + \Delta \Diamond (W(n, n)W(n, m)) \\ & + \sum_{k=1}^{n-1} \Delta\, W(k, n)(D_2^+ D_2^- W(k, m)) - \sum_{k=1}^{n-1} \Delta\, (D_2^+ D_2^- W(k, n))W(k, m) \\ & - D_n^-(W(n, n)W(n, m)) - (D_2^+ W(n, n))W(n, m) \end{aligned} \tag{5.4}$$

Now

$$D_n^-(W(n, n)W(n, m)) = (D_n^- W(n, n))W(n, m) + W(n - 1, n - 1)(D_1^- W(n, m))$$

So (5.4) can be rewritten as

$$\diamond W(n, m) + \sum_{k=1}^{n-1} \Delta W(k, n)(D_2^+ D_2^- W(k, m)) - \sum_{k=1}^{n-1} \Delta(D_2^+ D_2^- W(k, n))W(k, m)$$

$$- (D_n^- W(n, n) + D_2^+ W(n, n))W(n, m) = -\Delta \diamond (W(n, n)W(n, m)) + W(n-1, n-1)D_1^- W(n, m) \tag{5.5}$$

Also formula 5 of section 3 implies

$$\sum_{k=1}^{n-1} \Delta(D_1^- D_1^+ W(k, n))W(k, m) = \sum_{k=1}^{n-1} \Delta W(k, n)(D_1^- D_1^+ W(k, m)) + (D_1^- W(n, n))W(n, m)$$

$$- W(n, n)D_1^- W(n, m) - (D_1^- W(1, n))W(1, m) + W(1, n)D_1^- W(1, m) \tag{5.6}$$

We shall examine the last two summands in the above more closely. For $n = 1 < m$, (5.1) reads

$$G(1, m) = \frac{1}{2}(F(m+1) + F(m-1)) = (1 + \Delta W(1, 1))W(1, m) \tag{5.7}$$

and for $n = m = 1$

$$\frac{1}{2}(F(2) + F(0)) = W(1, 1) + \Delta W(1, 1)^2$$

It follows that

$$W(1, 1) = F(0) + O_1(\Delta) = H + O_1(\Delta)$$

where the bound x in the O_1 statement can be estimated in terms of $\|F\|_\infty$, $\|DF\|_\infty$. On the other hand,

$$\frac{1}{2}(F(m+1) + F(m-1)) = F(m) + \frac{\Delta^2}{2} D^+ D^- F(m)$$

Combine these facts to obtain

$$W(1, m) = (1 - H\Delta)F(m) + O_2(\Delta)$$

where the bound in O_2 can be estimated in terms of $\|F\|_\infty$, $\|DF\|_\infty$, $\|D^+ D^- F\|_\infty$. It follows (recalling that $W(0, m) = F(m)$) that

$$D^- W_1(1, m) = -H F(m) + O_1(\Delta) \tag{5.8}$$

and finally that

$$(D^- W(1, n))W(1, m) - W(1, n)D_1^- W(1, m) = -H F(n)(1 - H\Delta)F(m)$$

$$+ H F(m)(1 - H\Delta)F(n) + 0_1(\Delta) \equiv -\Delta R_1(n, m, \Delta)$$

where, for Δ sufficiently small, $\|R_1\|_\infty$ can be estimated in terms of $\|F\|_\infty$, $\|DF\|_\infty$, and $\|D^+ D^- F\|_\infty$.

Now rewrite (5.6) as

$$\sum_{k=1}^{n-1} \Delta(D_1^- D_1^+ W(k, n))W(k, m) - \sum_{k=1}^{n-1} \Delta W(k, n)(D_1^- D_1^+ W(k, m))$$
$$- (D_1^- W(n, n))W(n, m) = -W(n, n)D_1^- W(n, m) + \Delta R_1(n, m, \Delta) \tag{5.9}$$

Add this identity to (5.5) to obtain

$$\Diamond W(n, m) + \sum_{k=1}^{n-1} \Delta W(k, n)\Diamond W(k, m) - \sum_{k=1}^{n-1} \Delta(\Diamond W(k, n))W(k, m)$$
$$+ V(n)W(n, m) = \Delta R(n, m, \Delta) \tag{5.10}$$

$$V(n) = -(D_n^- + D_2^+ + D_1^-)W(n, n)$$

where

$$R(n, m, \Delta) = -\Diamond[W(n, n)W(n, m)] - D_n^- W(n, n)D_1^- W(n, m) + R_1(n, m, \Delta) \tag{5.11}$$

According to the results of section 4, $\|R\|_\infty$ may be estimated, for small Δ, in terms of $\|F\|_\infty$, $\|DF\|_\infty$, $\|D^+ D^- F\|_\infty$.

Notice that we can replace the factor $\Diamond W(k, m)$ in the first sum on the L.H.S. of (5.10) by $\Diamond W(k, m) + V(k)W(k, m)$, provided we also replace $\Diamond W(k, n)$ in the second sum by $\Diamond W(k, n) + V(k)W(k, n)$.

Define

$$U(n, m) = \Diamond W(n, m) + V(n)W(n, m) \quad , \qquad 1 \leq n < m$$

Then U obeys the discrete Volterra equation

$$U(n, m) + \sum_{k=1}^{n-1} \Delta W(k, n)U(k, m) - \sum_{k=1}^{n-1} \Delta U(k, n)W(k, m) = \Delta R(n, m, \Delta) \tag{5.12}$$

According to Lemma 3.3, the solution enjoys the same sort of bounds as the inhomogeneous term. In particular, we get $U = \Delta P$, so that

$$\Diamond W(n, m) + V(n)W(n, m) = \Delta P(n, m, \Delta) \tag{5.13}$$

where $\|P\|_\infty$ is bounded, uniformly for small Δ, in terms of $\|F\|_\infty$, $\|DF\|_\infty$, $\|D^+ D^- F\|_\infty$, and $\|W\|_\infty$, hence (according to the results of section 4) in terms of $\|F\|_\infty$, $\|DF\|_\infty$, $\|D^+ D^- F\|_\infty$, and ϵ^{-1}.

For future reference, we collect (5.13), (5.11), and (5.8) in the form of a boundary-value problem for W:

$$\Diamond W + VW = \Delta P \tag{5.14a}$$

$$V(n) = -(D_n^- + D_2^+ + D_1^-)W(n, n) \tag{5.14b}$$

$$\begin{aligned} &W(0, m) = F(m) \\ &D_1^+ W(0, m) + H\, W(0, m) = \Delta B(m, \Delta) \end{aligned} \tag{5.14c}$$

where in the last line of (5.14c), the vector $B(m, \Delta)$, $M \geq 0$, defined implicitly in the foregoing, is bounded uniformly for small Δ in terms of $\|F\|_\infty$, $\|DF\|_\infty$, and $\|D^+ D^- F\|_\infty$.

§6. Estimates for Chudov Schemes

The purpose of the present section is to compare the solution W of the Cholesky equation (5.1), (alternatively, of the boundary-value problem (5.14)), to the solution W_0 of the first order Chudov Scheme (equations (2.10) of section 2), which we display here for convenience:

$$\diamond W_0 + V_0 W_0 = 0 \tag{6.1a}$$

$$V_0(n) = -(D_n^- + D_2^+ + D_1^-)W_0(n, n) \tag{6.1b}$$

$$\begin{aligned} &W_0(0, m) = F(m) \\ &D^+ W_0(0, m) + H W_0(0, m) = 0 \\ &H = F(0) \end{aligned} \tag{6.1c}$$

The following notational convention is convenient: for any gridfunction $A(n, m)$ defined for $n \leq m$, set

$$\tilde{D} A(n, n) \equiv -(D_n^- + D_2^+ + D_1^-)A(n, n)$$

Then (6.1a), (6.1b) together become

$$\diamond W_0(n, m) + W_0(n, m)\tilde{D} W_0(n, n) = 0 \tag{6.2}$$

Similarly (5.14a), (5.14b) become

$$\diamond W(n, m) + W(n, m)\tilde{D} W(n, n) = \Delta P(n, m, \Delta) \tag{6.3}$$

Subtracting (6.2) from (6.3) we obtain for the difference $\bar{W} = W - W_0$

$$\diamond \bar{W}(n, m) + V(n)\bar{W}(n, n) + W(n, m)\tilde{D} \bar{W}(n, n) - \bar{W}(n, m)\tilde{D} \bar{W}(n, n) = \Delta P(n, m, \Delta) \tag{6.4}$$

$\bar{W}$ also satisfies the boundary equations

$$\begin{aligned} &\bar{W}(0, m) \equiv 0 \\ &D_1^+ \bar{W}(0, m) + H \bar{W}(0, m) = \Delta B(m, \Delta) \end{aligned} \tag{6.5}$$

We shall show that both $\bar{W}$ and its first differences are $O(\Delta)$ as $\Delta \to 0$, in the domain

$$C_T^\Delta = \{(n, m) : 0 \leq n \leq \min (m, 2N - m)\}$$

where $N = T/\Delta$.

The main tool is the next lemma, which shows that bounds of the type we want can be propagated to finite "depth" ($n\Delta \simeq$ const.), with controlled growth in the constants of proportionality.

<u>Lemma</u> (6.1) Suppose that $\tilde{W}$ satisfies the difference equation

$$\Diamond\, \tilde{W}(n, m) + V(n)\tilde{W}(n, m) + W(n, m)\tilde{D}\, \tilde{W}(n, n) - \tilde{W}(n, m)\bar{D}\, \tilde{W}(n, n) = \Delta\, P(n, m, \Delta) \quad , \qquad n \geq 1$$

in the half-lattice $\{(n, m) : n \geq 0\}$. Suppose in addition that the following estimates hold:

$$|\tilde{W}(0, m)| \quad , \quad |\tilde{W}(1, m)| \leq \Delta\, C_0$$

$$|\tilde{W}(1, m) - \tilde{W}(0, m - 1)| \leq \Delta^2\, C_1$$

for $m \in \mathbb{Z}$, and

$$|W(n, m)| \leq C_3 \quad , \quad |V(n)| \leq C_4$$

$$|P(n, m)| \leq C_5$$

for $n \geq 0$, $m \in \mathbb{Z}$. Then for $\kappa > 1$ $\tilde{W}$ satisfies

$$|\tilde{W}(n, m)| \leq \Delta\, \kappa\, C_0$$

$$|\tilde{W}(n, m) - \tilde{W}(n - 1, m - 1)| \leq \Delta^2\, \kappa\, C_1$$

provided $n\Delta = \tau$ satisfies $\tau < \delta$, where $\delta = \delta(\kappa, C_1, C_2, C_3, C_4, C_5, \Delta) > 0$ is bounded away from zero, for fixed $\kappa > 1$, $C_1, \ldots, C_5$ independently of $\Delta \to 0$.

<u>Proof</u>: You first establish the representation

$$\tilde{W}(n, m) = \sum_{j=0}^{n-1} \tilde{W}(1, n + m - 2j - 1) - \sum_{j=1}^{n-1} \tilde{W}(0, n + m - 2j) + \sum_{C(n,m)} \Delta^3\, Q(k, j) \qquad (6.6)$$

$$n \geq 1$$

where $C(n, m) = \{(k, j) : 0 \leq k \leq n,\ m - n + k \leq j \leq m + n - k\}$ and $Q(0, m) = Q(1, m) \equiv 0$.

The representation (6.6) clearly holds for $n = 1$. Assume that it holds for $W(k, m)$, $m \in \mathbb{Z}$ and $k < n$. Write the difference equation in the form

$$\bar{W}(n, m) = \bar{W}(n - 1, m + 1) + \bar{W}(n - 1, m - 1) - \bar{W}(n - 2, m)$$
$$+ \Delta^2 V(n - 1)\bar{W}(n - 1, m) + \Delta^2 W(n - 1, m)\tilde{D}\,\bar{W}(n - 1, n - 1)$$
$$- \Delta^2 \bar{W}(n - 1, m)\tilde{D}\,\bar{W}(n - 1, n - 1) - \Delta^3 P(n, m)$$

Note that $(C(n - 1, m + 1) \cup C(n - 1, m - 1)) \setminus C(n - 2, m) = C(n, m) \setminus \{(n, m)\}$.

Also,

$$\begin{aligned}
&\sum_{j=0}^{n-2} \bar{W}(1, n + m - 2j - 1) - \sum_{j=1}^{n-1} \bar{W}(0, n + m - 2j) \\
&+ \sum_{j=0}^{n-2} \bar{W}(1, n + m - 2j - 3) - \sum_{j=1}^{n-1} \bar{W}(0, n + m - 2j - 2) \\
&- \sum_{j=0}^{n-3} \bar{W}(1, n + m - 2j - 3) + \sum_{j=1}^{n-3} \bar{W}(0, n + m - 2j - 2) \\
&= \sum_{j=0}^{n} \bar{W}(1, n + m - 2j - 1) - \sum_{j=1}^{n} \bar{W}(0, n + m - 2j)
\end{aligned} \tag{6.7}$$

Obtain from (6.4), (6.7) and the induction hypothesis

$$\bar{W}(n, m) = \sum_{j=0}^{n} \bar{W}(1, n + m - 2j - 1) - \sum_{j=\phi}^{n} \bar{W}(0, n + m - 2j)$$
$$+ \sum_{C(n,m)\setminus\{(n,m)\}} \Delta^3 Q(k, j) + \Delta^2 V(n - 1)\bar{W}(n - 1, m)$$
$$+ \Delta^2 W(n - 1, m)\tilde{D}\ \bar{W}(n - 1, n - 1) - \Delta^2 \bar{W}(n - 1, m)\tilde{D}\,\bar{W}(n - 1, n - 1) - \Delta^3 P(n, m)$$

The representation formula therefore holds for (n, m) if you set

$$\begin{aligned}
Q(n, m) = -P(n, m) + \Delta^{-1}\{&V(n - 1)\bar{W}(n - 1, m) + \\
&W(n - 1, m)\tilde{D}\,\bar{W}(n - 1, n - 1) - \bar{W}(n - 1, m)\tilde{D}\,\bar{W}(n - 1, n - 1)\}
\end{aligned} \tag{6.8}$$

Denote by $\bar{Q}(n) = \sup\limits_{\substack{k<n \\ m\in\mathbb{Z}}} |Q(k, m)|$. The next step is to estimate $\bar{Q}(n)$ in terms of $\bar{Q}(n-1)$, C_μ, $\mu = 1, \ldots, 5$.

The main ingredient is the estimation of $\tilde{D}\,\bar{W}$. Recall that $\tilde{D} = D_n^- + D_2^+ + D_1^-$.

$$D_n^- \bar{W}(k, k) = \Delta^{-1}(\bar{W}(k, k) - \bar{W}(k-1, k-1)) = \Delta^{-1}[\bar{W}(1, 2k-1)$$

$$- \bar{W}(0, 2k-2) + \sum_{j=0}^{k} \Delta^3(Q(k-j, k+j) + Q(k-j-1, k+j))]$$

So $|D_n^- W(k, k)| \le \Delta\, C_1 + 2k\,\Delta^2\,\bar{Q}(k)$ (6.9a)

Similarly

$$|(D_2^+ + D_1^-)W(k, k)| \le \Delta\, C_1 + 2k\,\Delta^2\,\bar{Q}(k) \tag{6.9b}$$

These estimates, when combined with (6.8) and the hypotheses, yield the required estimate of $\bar{Q}(n)$.

The remainder of the proof is a finite induction argument.

<u>Claims</u>: Suppose

$$\tau = n\Delta \le \min\left\{(8(C_3 + \Delta\,\kappa\, C_0))^{-1},\ (\kappa - 1)\frac{C_1}{2\bar{Q}},\ \left[\left(\frac{C_1}{\bar{Q}}\right)^2 + 2\,\frac{(\kappa-1)C_0}{\bar{Q}}\right]^{1/2} - \frac{C_1}{\bar{Q}}\right\}$$

Then

(i) $|Q(n, m)| \le \bar{Q} \equiv 2[C_5 + 2\,C_1\,C_3 + \kappa(C_0\,C_4 + 2\,\Delta\,C_0\,C_1)]$

(ii) $|\bar{W}(n, m)| \le \Delta\,\kappa\,C_0$

(iii) $|\bar{W}(n, m) - \bar{W}(n-1, m-1)| \le \Delta^2\,\kappa\,C_1$

For $n = 1$, (i) holds since $Q(1, \cdot) = 0$; (ii) and (iii) are just the hypotheses of the Lemma. Suppose (i), (ii), (iii) hold for $n \le \bar{n} - 1$. Then from (6.8), (6.9) you obtain

$$|Q(\bar{n}, m)| \le C_5 + \Delta^{-1}\{C_4\,\Delta\,\kappa\,C_0 + 2\,C_3\,\Delta\,C_1 + 4\,C_3\,\Delta^2(\bar{n}-1)\bar{Q} + \Delta\,\kappa\,C_0(2\,\Delta\,C_1 + 4(\bar{n}-1)\Delta^2\,\bar{Q})\}$$

$$\le C_5 + \kappa\,C_0\,C_4 + 2\,C_1\,C_3 + 4\,C_3\,\Delta\,\bar{n}\,\bar{Q} + \Delta\,\kappa\,C_0(2\,C_1 + 4\,\bar{n}\,\Delta\,\bar{Q})$$

$$= C_5 + 2\,C_1\,C_3 + \kappa(C_0\,C_4 + 2\,\Delta\,C_0\,C_1) + 4\,\bar{n}\,\Delta(C_3 + \Delta\,\kappa\,C_0)\bar{Q}$$

The hypothesis of the claim is that $\bar{\tau} = \bar{n}\Delta \leq (8(C_3 + \Delta\,\kappa\,C_0))^{-1}$ among other things. Together with the definition (i) of $\bar{Q}$, this implies that the above is

$$\leq \frac{1}{2}\,\bar{Q} + \frac{1}{2}\,\bar{Q} = \bar{Q}$$

which establishes (i).

The second part of the hypothesis on $\bar{\tau}$, namely

$$\bar{\tau} \leq \left\{\left(\frac{C_1}{\bar{Q}}\right)^2 + 2\,\frac{(\kappa - 1)C_0}{\bar{Q}}\right\}^{1/2} - \frac{C_1}{\bar{Q}}$$

means exactly that

$$0 \leq \tau^2\,\frac{\bar{Q}}{2} + \tau\,C_1 - (\kappa - 1)C_0 \tag{6.10}$$

On the other hand, the representation (6.6), estimated according to (i) and the hypotheses of the Lemma, gives

$$|\bar{W}(\bar{n}, m)| = \Big|\sum_{j=0}^{n-1} \{\bar{W}(1, n + m - 2j - 1) - \bar{W}(0, n + m - 2j - 2)\}$$

$$+ \bar{W}(0, m - n) + \sum_{C(n,m)} \Delta^3\,Q(k, j)\Big| \leq \bar{n}\,\Delta^2\,C_1 + \Delta\,C_0 + \frac{\bar{n}^2\,\Delta^3}{2}\bar{Q} = \Delta(C_0 + \tau\,C_1 + \tau^2\,\frac{\bar{Q}}{2})$$

which is

$$\leq \Delta\,\kappa\,C_0$$

if and only if (6.10) holds. Finally, using (6.6) as before, you show that

$$|W(\bar{n}, m) - W(\bar{n} - 1, m - 1)| \leq \Delta^2\,C_1 + 2\,\Delta^3\,\bar{n}\,\bar{Q} = \Delta^2\,C_1(1 + 2\,\Delta\,\bar{n}\,\frac{\bar{Q}}{C_1})$$

$$\leq \Delta^2\,C_1(1 + 2(\kappa - 1)\,\frac{C_1}{2\bar{Q}} \cdot \frac{\bar{Q}}{C_1}) = \Delta^2\,\kappa\,C_1\ .$$

The claim is thus proven, and with it the Lemma, if you take

$$\delta = \delta(\kappa, C_0, C_1, C_2, C_3, C_4, C_5, \Delta) \tag{6.11}$$

$$= \min\left\{\frac{1}{8(C_3 + \Delta\kappa C_0)}, \frac{(\kappa - 1)C_1}{2\bar{Q}}, \left[\left(\frac{C_1}{\bar{Q}}\right)^2 + 2\frac{(\kappa - 1)C_0}{\bar{Q}}\right]^{1/2} - \frac{C_1}{\bar{Q}}\right\}$$

For fixed $\kappa > 1$ and $C_0 - C_5$, δ is uniformly bounded away from zero as $\Delta \to 0$, so satisfies the requirements of the Lemma.

q.e.d.

Remark. The above specification of δ allows one to determine an optimal choice of $\kappa > 1$, so as to make δ as large as possible for given $C_0 - C_5$.

Recall that $N = \Delta^{-1} T$.

Corollary 6.2. Same hypotheses as in Lemma 6.1.

There exists $\sigma > 0$ so that, for Δ sufficiently small

$$\sup_{\substack{0 \le n \le N \\ m \in \mathbb{Z}}} |\bar{W}(n, m)| \le \Delta \kappa^\sigma C_0 \le C_0$$

and

$$\sup_{\substack{0 \le n \le N \\ m \in \mathbb{Z}}} |\bar{W}(n, m) - \bar{W}(n - 1, m - 1)| \le \Delta^2 \kappa^\sigma C_1$$

Proof. The hypotheses and conclusion of the lemma are so designed that it can be applied iteratively to show estimates of the following form:

$$|\bar{W}(n, m)| \le \Delta\kappa C_0 \quad \text{for} \quad 0 \le n\Delta \le \delta_1 = \delta(C_0, C_1, C_2, C_3, C_4, C_5, \kappa)$$

$$|\bar{W}(n, m)| \le \Delta\kappa^2 C_0 \text{ for } \delta_1 \le n\Delta \le \delta_1 + \delta_2, \ \delta_2 = \delta(\kappa C_0, \kappa C_1, C_2, C_3, C_4 \cdot C_5, \kappa) \cdots$$

$$|\bar{W}(n, m)| \le \Delta\kappa^j C_0 \quad \text{for} \quad \delta_1 + \cdots + \delta_{j-1} \le n\Delta \le \delta_1 + \cdots + \delta_j \quad \text{with}$$

$$\delta_j = \delta(\kappa^{j-1} C_0, \kappa^{j-1} C_1, C_2, C_3, C_4, C_5, \kappa)$$

Set

$$\bar{Q}_0 = 2\,C_5 + 2\,C_1\,C_3 + \quad C_0(C_4 + 2\,C_1)$$

$$\delta_0 = \delta_0(\kappa,\,C_0,\,C_1,\,C_2,\,C_3,\,C_4,\,C_5)$$

$$\equiv \min\left\{\frac{1}{8(C_3 + C_0)}\ ,\ \frac{(\kappa - 1)C_1}{\bar{Q}_0}\ ,\ \left[\left(\frac{C_1}{\bar{Q}_0}\right)^2 + \frac{2(\kappa - 1)C_0}{\bar{Q}_0}\right]^{1/2} - \frac{C_1}{\bar{Q}_0}\right\}$$

and define $\sigma = T/\delta_0$. Suppose that Δ is so small that $\Delta\,\kappa^{\sigma} \le 1$. Then for $1 \le j \le \sigma$ certainly

$$8(C_3 + \Delta\,\kappa^j\,C_0) \le 8(C_3 + C_0)$$

Also

$$\bar{Q}_j \equiv \bar{Q}(\kappa,\,\kappa^j\,C_0,\,\kappa^j\,C_1,\,C_2,\,C_3,\,C_4,\,C_5) = 2\,C_5 + 2\,\kappa^j\,C_1\,C_3 + (\kappa^j\,C_0\,C_4 + 2\,\Delta\,\kappa^{2j}\,C_0\,C_1)$$

$$\le 2\,C_5 + 2\,\kappa^j\,C_1\,C_3 + \kappa^{j+1}\,C_0(C_4 + 2\,C_1) \le \kappa^j(2\,C_5 + 2\,C_1\,C_3 + \kappa\,C_0(C_4 + 2\,C_1)) = \kappa^j\,\bar{Q}_0$$

Hence, if

$$\tau \le \left[\left(\frac{C_1}{\bar{Q}_0}\right)^2 + 2\,\frac{(\kappa - 1)C_0}{\bar{Q}_0}\right]^{1/2} - \frac{C_1}{\bar{Q}_0}$$

that is

$$0 \le \tau^2\,\frac{\bar{Q}_0}{2} + \tau\,C_1 - (\kappa - 1)C_0$$

then

$$0 \le \tau^2\,\kappa^j\,\frac{\bar{Q}_0}{2} + \tau\ {}^{j}\,C_1 - (\kappa - 1)\ {}^{j}\,C_0$$

$$\le \tau^2\,\frac{\bar{Q}_j}{2} + \tau\,\kappa^j\,C_1 - (\kappa - 1)\ {}^{j}\,C_0$$

hence

$$\tau \le \left[\left(\frac{\kappa^j\,C_1}{\bar{Q}_j}\right)^2 + 2\,\frac{(\kappa - 1)\ {}^{j}\,C_0}{\bar{Q}_j}\right]^{1/2} - \frac{\kappa^j\,C_1}{\bar{Q}_j}$$

Referring to the definition (6.11) of δ, you see that, $\delta_j \geq \delta_0$, $j = 1, \ldots, \sigma$. So you can replace δ_j by δ_0 in the estimates at the beginning of the proof to get

$$|\bar{W}(n, m)| \leq \Delta \kappa^j C_0$$

for $(j - 1)\delta_0 \leq n\Delta \leq j \delta_0$, $j = 1, \ldots, \sigma$. For $j = \sigma$, this gives the estimate we want for $\bar{W}$. The second estimate now follows in exactly the same way.

q.e.d.

Theorem 6.3. Suppose $\bar{W}$ satisfies (6.4), (6.5). Then for $(n, m) \in C_T^\Delta$, arbitrary $\kappa > 1$, and Δ sufficiently small,

$$|\bar{W}(n, m)| \leq \Delta \kappa^\sigma C_0$$

$$|\tilde{D} \bar{W}(n, m)| \leq \Delta \kappa^\sigma C_1$$

where C_0, C_1 are entire functions of ϵ^{-1}, Δ, T, $\|F\|$, $\|D^\pm F\|$, $\|D^+ D^- F\|$ and $\sigma > 0$ is the minimum envelope of entire functions of these arguments.

Proof. First observe that the difference scheme (6.4) restricts to C_T^Δ. Also, all of the constructions used in the proofs of Lemma 6.1 and Corollary 6.2 restrict to C_T^Δ.

Now (6.5) can be written

$$\bar{W}_1(1, m) = \Delta^2 B(m, \Delta)$$

It follows that the hypotheses of Lemma 6.1 are satisfied, with C_1 estimated by the bound for B mentioned at the end of section 5, $C_0 = \Delta C_1$, C_5 estimated by the bound for P given in section 5, and C_3 and C_4 estimated by the bounds for W, V given in section 4. Note that

$$\begin{aligned}
\tilde{D} \bar{W}(n, n) &= -(D_n^- + D_2^+ + D_1^-)W(n, n) \\
&= -\Delta^{-1} \{W(n, n) - W(n - 1, n - 1) + W(n, n + 1) - W(n, n) \\
&\qquad + W(n, n) - W(n - 1, n)\} \\
&= -\Delta^{-1} \{W(n, n) - W(n - 1, n - 1) + W(n, n + 1) - W(n - 1, n)\}
\end{aligned}$$

and both of the differences in this expression are estimated by <u>Corollary</u> 6.2 which leads to the second inequality in the statement. Note that the definition of δ_0 in the proof of <u>Corollary</u> 6.2 shows that σ is as described.

q.e.d.

<u>Corollary</u> 6.4. For the solution W_0, V_0 of (6.1) and the solution W, V of (5.14), we have: for any $\rho > 0$, there is $\Delta_0 > 0$ so that for $0 < \Delta \leq \Delta_0$,

$$\|W_0\| \leq (1 + \rho)\|W\|$$

$$\|V_0\| \leq (1 + \rho)\|V\|$$

We now turn to the second-order consistent scheme mentioned in section 2, which we reproduce here:

$$\Diamond W_1 + V_1 W_1 = 0 \tag{6.12a}$$

$$\begin{aligned} &W_1(0, m) = F(m) \\ &W_1(1, m) = (\tfrac{1}{2} V_1(0)\Delta^2 - H)W_1(0, m) + \tfrac{1}{2}(W_1(0, m+1) + W_1(0, m-1)) \end{aligned} \tag{6.12b}$$

$$\begin{aligned} V_1(n) &= -(D_n^+ + D_n^-)W_1(n, n) \\ H &= F(0) \end{aligned} \tag{6.12c}$$

As it stands, this scheme is incomplete: the value $V_1(0)$ is not determined from the data. In order to preserve second-order consistency we add the following one-sided difference approximation to $V_1(0)$, obtained from the GL equation and the boundary condition (2.2c):

$$V_1(0) = 2H^2 - 2\Delta^{-1}(2F(1) - \tfrac{1}{2}F(2) - \tfrac{3}{2}F(0)) \tag{6.12d}$$

<u>Theorem</u> 6.5. For any $\rho > 0$, there exists $\Delta_1 > 0$ so that for $0 < \Delta \leq \Delta_1$, the solution W_1 of (6.12a - d) satisfies

$$\|W_1\| \leq (1 + \rho)\|W\|$$

$$\|V_1\| \leq (1 + \rho)\|V\|$$

Proof. The idea is to compare W_1 and W_0 in the same way W_0 and W were compared above.

Write

$$D' \, W_\nu(n, n) = -(D_n^+ + D_n^-) W_\nu(n, n) \quad , \qquad \nu = 0, 1$$

Then (6.1) can be rewritten

$$\diamond \, W_0(n, m) + W_0(n, m) D' \, W_0(n, n) = \Delta \, P_1(n, m, \Delta) \qquad (n, m) \in C_T^\Delta \tag{6.13a}$$

$$W_0(0, m) = F(m)$$

$$W_0(1, m) = (\tfrac{1}{2} V(0)\Delta^2 - H\Delta) W_0(0, m) + (1 - \tfrac{1}{2} V(0)\Delta^2) W_0(0, m) \tag{6.13b}$$

Here

$$P_1(n, m, \Delta) = \Delta^{-1}(D' \, W_0(n, n) - \tilde{D} \, W_0(n, n)) W_0(n, m)$$

$$= \Delta^{-1}(D_n^+ \, W_0(n, n) - D_2^+ \, W_0(n, n) - D_1^- \, W_0(n, n)) W_0(n, m)$$

$$= \Delta^{-1}(D_1^+ \, W_0(n, n + 1) - D_1^- \, W_0(n, n)) W_0(n, m)$$

$$= \Delta^{-1}(\Delta \, D_1^+ \, D_1^- \, W_0(n, n + 1) + D_1^- \, W_0(n, n + 1) - D_1^- \, W_0(n, n)) W_0(n, m)$$

$$= (D_1^+ \, D_1^- \, W_0(n, n + 1) + D_2^+ \, D_1^- \, W_0(n, n)) W_0(n, m)$$

According to Proposition 4.3(c), P_1 can be estimated by an entire function of ϵ^{-1}, T, Δ, and norms of F and its first and second differences.

For the difference $W' = W_0 - W_1$ we obtain

$$\diamond \, W'(n, m) + V_0(n) W'(n, m) + W_0(n, m) D' \, W'(n, n)$$

$$- W'(n, m) D' \, W'(n, n) = \Delta \, P_1(n, m, \Delta) \qquad (n, m) \in C_T^\Delta \tag{6.14a}$$

$$W'(0, m) \equiv 0$$
$$W'(1, m) \equiv \Delta^2 B_1(m, \Delta) \tag{6.14b}$$

Here

$$B_1(m, \Delta) = \frac{1}{2} D^+ D^- F(m) - \frac{1}{2} V(0)F(m)$$

From this point, the proof follows exactly the proofs of Lemma 6.1, Corollary 6.2, Theorem 6.3, and Corollary 6.4, and we leave the details to the reader. We note in passing that estimates appear for W' and $D' W'$ analogous to those of Theorem 6.3 for $\bar{W}$ and $\tilde{D} \bar{W}$.

q.e.d.

Remark. It seems likely that similar results could be obtained for higher-order consistent schemes for the Chudov problem.

§7. Results of Numerical Experiments

We present the results of two series of numerical experiments carried out at the University of Wisconsin - Madison MACC facility. The computations were performed in single precision on the UNIVAC 1110.

Both series of experiments involved a FORTRAN program, detailed below, which implements the second-order Chudov Scheme (6.12). In both cases V is to be computed on $[0, 1]$, i.e. $T = 1$. The series differ in the mode of data generation.

In the first series, the inverse problem for $F(t) \equiv F$ (const.) is solved by way of (6.12). The approximate potential $V(n) = V(n\Delta)$ is compared with the exact potential $VE(n) = VE(n\Delta)$, which is known in closed analytic form:

$$\begin{aligned} H &= F(0) \\ VE(x) &= \frac{2}{\left(x + \frac{1}{H}\right)^2} \end{aligned} \tag{7.1}$$

Both the maximum (sup norm) error

$$\sup_{1 \le n \le N} |V(n - 1) - VE(n - 1)|$$

and the average $(L^1 -)$ error

$$\frac{1}{N} \sum_{n=1}^{N} |V(n - 1) - VE(n - 1)|$$

are displayed.

The last three experiments in series 1 are meant to illustrate the dependence of the error on the lower bound ϵ, which for these examples is equal to $1 + H = 1 + F$. As $F \to -1$, the exact potential given by (7.1) clearly admits no uniform Lipschitz bound in terms of norms of F alone (in fact, the analogous expression for W shows that the bound (2.5a) is sharp). This lack of Lipschitz uniformity is reflected in the behaviour of the computation, as predicted by the theory. For $H = -.99$, the computer produces garbage for $N = 11$. Even for $N = 101$, the errors are relatively large, although almost all of the error is concentrated at the "deepest" end of the interval (near $x = 1$ - an effect also predicted by the theory).

The second series of experiments is based on numerically computed F for various V, H. That is, VE and H are selected, the Chudov problem is solved numerically for F (a version of the forward scattering problem), then the program given below is executed with various samplings of this numerically-generated F as input, and the resulting approximate V is compared with VE. Again, maximum and average errors are displayed.

The main difficulty in this second series was the data (F -) generating program, which was asked in effect to solve a discontinuous initial value problem for a wave equation. We finally settled on a staggered-grid method which computes F at 2000 points on the t-axis, which are then sampled to produce the input for the inverse program. Despite the relatively fine grid of the forward computation, error in F has an observable effect on the convergence of the inversion scheme in several examples, in the $(N = 51)$ to $(N = 101)$ step.

Except as just noted, quadratic convergence is observed in all experiments.

It is difficult to assess the error figures given in the tables below, since the problem solved here is merely a model problem, with no physical interpretation whatsoever attached. For the same reason, we have not bothered to predict the errors based on the theory. Both of these items would be interesting for some of the problems mentioned at the end of the next section.

Program Implementing (6.12)

1. Read VE(I) , I = 1 , N

   F(I) , I = 1 , 2N - 1

2. H = F(1)

3. $V(1) = 2\ H^2 - 2\ \Delta^{-1}(2\ F(2) - \frac{1}{2} F(3) - \frac{3}{2} F(1))$

4. For I = 1 , 2N - 1

   W(1, I) = F(I)

5. For I = 2 , 2N - 2

   $W(2, I) = (\frac{1}{2}\ \Delta\ V(1) - H)W(1, I) + \frac{1}{2}\ (W(1, I + 1) + W(1, I - 1))$

6. For I = 3 , N

   6.1 $W(I, I) = W(I - 1, I + 1) + W(I - 1, I - 1) - W(I - 2, I)$

   $+ \Delta[W(I - 2 , I - 2)W(I - 1, I)]\ [1 + \Delta\ W(I - 1, I)]^{-1}$

   6.2 $V(I - 1) = -\Delta^{-1}(W(I, I) - W(I - 2, I - 2))$

   6.3 For J = I + 1 , M - I + 1

   6.3.1 $W(I, J) = W(I - 1, J + 1) + W(I - 1, J - 1) - W(I - 2, J)$

   $+ \Delta^2\ V(I - 1)W(I - 1, J)$

7. $\text{MAX ERROR} = \max_{1 \le I, \le N-1} |V(I) - VE(I)|$

8. $\text{AVE ERROR} = (N - 1)^{-1} \sum_{I=1}^{N-1} |V(I) - VE(I)|$

9. END

TABLE I

$$V(x) = \frac{2}{\left(x + \frac{1}{H}\right)^2} \qquad F(t) \equiv H$$

		DELTA	MAX ERROR		AVG. ERROR	
H = .1	11	.1	.0		.0	
	21	.05	.0		.0	
	51	.02	.0		.0	
	101	.01	.0		.0	
H = .5	11	.1	.4	-3	.27	-3
	21	.05	.1	-3	.7	-4
	5.	.02	.2	-4	.1	-4
	101	.01	.1	-4	.1	-4
H = 2.	11	.1	.11		.93	-1
	21	.05	.28	-1	.24	-1
	51	.02	.45	-2	.41	-2
	101	.01	.12	-2	.10	-2
H = 5.	11	.1	.40	+1	.24	+1
	21	.05	.11	+1	.86	
	51	.02	.17		.16	
	101	.01	.44	-1	.41	-1
H = -.5	11	.1	.83	-2	.23	-2
	21	.05	.28	-2	.65	-3
	51	.02	.46	-3	.11	-3
	101	.01	.10	-3	.2	-4
H = -.9	11	.1	.11	+2	.14	+1
	21	.05	.73	+1	.58	
	51	.02	.26	+1	.13	
	101	.01	.86		.38	-1
H = -.99	11	.1	.54	+3	.55	+2
	21	.05	.11	+4	.57	+2
	51	.02	.17	+4	.38	+2
	101	.01	.16	+4	.20	+2

$\Delta = .01$: errors < 1 for $0 \le x \le .9$; errors < .1 for $0 \le x \le .68$. Also $V(1) \approx 2 \times 10^4$.

TABLE II

A. $V(X) \equiv 0$ $H = 2$

N	Δ	MAX. ERROR		AVG. ERROR	
11	.1	.1		.97	-1
21	.05	.30	-1	.18	-1
51	.02	.53	-2	.15	-2
101	.01	.18	-2	.57	-3

B. $V(X) \equiv 0$ $H = -2$

11	.1	.9	-1	.35	-1
21	.05	.24	-1	.11	-1
51	.02	.33	-2	.29	-2
101	.01	.15	-2	.14	-2

C. $V(X) = 1 - x$ $H = 0$

11	.1	.25	-2	.17	-2
21	.05	.61	-3	.41	-3
51	.02	.10	-3	.6	-4
101	.01	.30	-4	.1	-4

D. $V(X) = X(1 - X)$ $H = 0$

11	.1	.42	-2	.36	-2
21	.05	.11	-2	.95	-3
51	.02	.26	-3	.19	-3
101	.01	.15	-3	.90	-4

E. $V(X) = \cos(2X)$ $H = 0$

11	.1	.64	-2	.37	-2
21	.05	.17	-2	.93	-3
51	.02	.27	-3	.15	-3
101	.01	.7	-4	.5	-4

F. $V(X) = 5 \cos(10X)$ $H = 0$

11	.1	.77		.48	
21	.05	.21		.13	
51	.02	.34	-1	.21	-1
101	.01	.85	-2	.53	-2

§8. Conclusion

We have still to discuss the relation of our results to work of other authors, and the relative efficiency of our method for solving the inverse problem. Since the first point bears on the second, we begin with it.

Our results belong to the line of work begun by Gel'fand and Levitan in [3]. We have discussed the relation of previous work in this line to our results for the continuum inverse problem in [1], and we defer further discussion of the various approaches to the continuum problem to another place. We restrict our discussion here to approximate methods.

All but one or two of the authors writing on inverse problems in the spirit of Gel'fand and Levitan base their work on certan a linear integral equation. This is either the Gel'fand-Levitan integral equation or the Marchenko integral equation, depending on whether the incident waves are incident at the origin or at infinity (the latter is always the case for the whole-line inverse scattering problem). Also, the inverse problems are usually formulated in the frequency domain: that is, the scattering datum is either the spectral function (which is the Fourier transform of our $F(t)$: see [1], section 3), the (frequency-dependent) phase shift, or the (frequency-dependent) scattering matrix or reflection coefficient.

Among the authors whose work on approximate solutions to inverse problems fits into the framework just described are Case [9], Case and Kac [10], Case and Chiu [11], Ware and Aki [12], Berryman [13], and Green and Berryman [14].

P. Gopillaud [15] has given a slightly different (and faster) algorithm for the normal incidence elastic waves inverse problem for layered media. Ware and Aki [12], Greene and Berryman [13], and Berryman [14] develop this idea further and show that the Gopillaud and discrete Gel'fand-Levitan-Marchenko approaches are equivalent, in various senses.

Our work differs from all of the above mentioned work in many respects, the following three of which we believe most important:

1) We base all of our results on the *nonlinear Volterra equation* (GL *in section* 2) rather than the linear Gel'fand-Levitan or Marchenko equations. Though these are, in principle, equivalent, it seems difficult to extract the proper stability results from the linear equations.

2) Both the formulation of our inverse problem and its solution are time-dependent. It is clear from the results of [1] that at least some of the "frightful instability" attributed by Wheeler [16] to the Gel'fand-Levitan method enters by way of the conditionally convergent Fourier integrals required to pass between the time-domain and frequency-domain problems.

3) Most important for application and generalization is numerical stability. Some stability result is required to guarantee that each algorithm will actually *converge as* $\Delta \to 0$ *to the solution of the continuum problems*. None of the above authors seem to provide the necessary estimates to conclude convergence. To provide such estimates is exactly the point of the present work.

The Cholesky decomposition estimates of section 4 should also provide estimates for the solution of the discrete versions of the linear Gel'fand-Levitan integral equation, hence imply convergence of the various algorithms so constructed. The same should be true, with proper attention to the behaviour of the potential at infinity, for the discrete Marchenko equation approaches.

We conjecture that the Gopillaud algorithm is actually closely related to our Chudov scheme, hence should inherit the stability properties derived here.

The approaches to the inverse problem based on the various integral equations have costs proportional to N^4 or N^3, depending on implementation (N = number of linear gridpoints, as usual). The Gopillaud scheme (and modifications - [12], [13], and [14]) has cost proportional to N^2. The Chudov schemes investigated here also have cost proportional to N^2, hence seem to be optimally cheap amongst "exact" inversion methods.

In applications of inverse problems, for instance in exploration geophysics, physical chemistry, and ultrasound tomography, various "approximate" *inversion methods* are used. Indeed these problems involve more than one space dimension, and "exact" Gel'fand-Levitan-Marchenko methods have yet to be extended in any useful way to higher-dimensional problems. These approximate methods are based on well-known approximate solution methods for the relevant partial differential equations, for instance Born series approximation (formal perturbation series) or JWKB methods (geometric optics). The approximate solutions are then inverted to obtain (one hopes) approximate solutions to inverse scattering problems. One can also apply these methods, as detailed for instance in [17], Chapter XVI, to the simple inverse problem of this paper. One obtains schemes with cost proportional to N^2 (or, for the Born approximation, $N \log N$ if one employs the Fast Fourier Transform). Our Chudov algorithm therefore *has more or less the same cost* as these "approximate" methods. Of course, the Chudov algorithms also enjoy stability properties, as we have shown, which imply the possibility of rigorous error estimation. No such possibility is available for the "approximate" methods, to our knowledge.

We describe in conclusion some other problems to which our methods apply. The analytical details (results analogous to Theorems 1 and 2, and estimates (2.5a, b, c)) have been worked out in [18] for an inverse problem for the acoustic wave operator

$$\Box_C = \frac{\partial^2}{\partial t^2} - C^2(x)\,\frac{\partial^2}{\partial x^2}$$

and in [19] for some inverse problems for general hyperbolic systems in two variables of first order, with constant sound speeds. Numerical results should also follow by the techniques of this paper. In these problems, the incident waves are incident at some finite point in the (one-dimensional) medium. Inverse scattering problems, in which waves are incident at infinity, should also submit to roughly the same approach, although the technical details remain to be worked out. We point out that inverse problems for the wave operator $\Box_C$ typically are approached through dependent- and independent-variable transformations which convert it into the potential perturbation of $\Box_0$ treated in this paper ([12], [14]). Neither this possibility nor the Gel'fand-Levitan-Marchenko linear integral equation approach are available for the general hyperbolic first-order system with more than one variable (unknown) sound speed. We have succeeded in solving some inverse problems for such systems by combining the techniques of [18] and [19].

Finally we mention that the artifacts of our approach to inverse problems (GL equation, Chudov boundary-value problem) are also available for higher-dimensional inverse problems. We have not yet derived the necessary a priori estimates to proceed with such an extension of the theory, however.

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The main result of this paper is a stability theorem for a certain class of difference algorithms designed to give approximate solutions of a model inverse scattering problem in one dimension. This stability result guarantees the convergence of the approximate solutions to the exact solution of the problem as the grid of the difference scheme is refined. We present the results of numerical experiments based on one of these schemes, in which second-order convergence is observed. Furthermore the cost (that is, the dependence on N of the number of arithmetic operations required to compute the solution at N grid points) of the algorithms discussed below is essentially optimal.

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